On the Existence of Efficient Points in Locally Convex Spaces

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Abstract. We study the existence of efficient points in a locally convex space ordered by a convex cone. New conditions are imposed on the ordering cone such that for a set which is closed and bounded in the usual sense or with respect to the cone, the set of efficient points is nonempty and the domination property holds.

Key words. Multiobjective optimization, efficient point, domination property.

1. Introduction

Let E be a topological vector space and $K \subset E$ be a nonempty convex cone. For $x, y \in E$ we write $x \leqslant_K y$ if $y - x \in K$ and $x <_K y$ if $x \leqslant_K y$ and $x \neq y$. The relation $\leqslant_K$ is reflective, transitive and, if K is pointed (i.e. $l(K) := K \cap (-K) = \{0\}$), antisymmetric. Thus E is partially ordered by the cone K.

Given a nonempty subset $A \subset E$, we say that an element $x \in A$ is an efficient (or Pareto-minimal, or nominated) point of A with respect to K if $y \leqslant_K x$ for some $y \in A$ then $x \leqslant_K y$. The set of efficient points of A with respect to K is denoted by $Min(A|K)$. In the sequel, if no confusion occurs we shall write $\leqslant$ and omit "with respect to K" in the definition above. Note that if K is pointed, a point $a \in A$ is an efficient point if there is no $y \in A$, $y \neq x$ such that $y \leqslant x$ or, equivalently, if $A_x = \{x\}$, where $A_x := A \cap (x - K)$, a section of A at x.

Throughout the paper, R^n denotes the n-dimensional Euclide space and $R_+ \subset R^1$ is the set of nonnegative scalars. For a set A in E, clA and A^C stand for its closure and complementation, respectively.

Let us note that even for a finite set $A \subset R^2$, $Min(A|K)$ may be empty. However, Hartley [7] showed that $Min(A|K)$ is nonempty for any set A such that $A \cap (x - clK)$ is nonempty compact for some $x \in E$. When E is a topological vector space the nonemptiness of $Min(A|K)$ has been established by Corley [5] for the case when K is acute (i.e. clK is pointed), A is nonempty K-semicompact (that means every open cover of A of the form $\{(x_\alpha - clK)^C: x_\alpha \in A, \alpha \in \mathscr{L}\}$ has a finite subcover) and by Borwein [2, 3] for any compact set A. Sterna-Karwat defined in [16, 17] a largest class $\mathscr{C}$ of convex cones ensuring the existence of efficient points in compact sets: if E is a Hausdorff topological space, K belongs to $\mathscr{C}$ if for every closed vector subspace L of E, $K \cap L$ is a vector subspace whenever its closure $cl(K \cap L)$ is a vector subspace. Some general existence theorems in topological vector spaces have been obtained by Luc [10] under the conditions that K is correct (i.e. $clK + K\backslash l(K) \subset K$) and A has a K-complete section. Recall that a set $A \subset E$ is said to be K-complete if it has no cover of the form $\{(x_\alpha - clK)^C: \alpha \in \mathscr{L}\}$ with $\{x_\alpha\} \subset A$ being a decreasing net (i.e. $x_\alpha < x_\beta$ for each $\alpha, \beta \in \mathscr{L}, \beta < \alpha$). Any K-compact set (that is a set any cover of which of the form $U_\alpha + C$: $\alpha \in \mathscr{L}$, U_α are open admits a finite subcover), in particular, any K-semicompact set or compact set is K-complete.

Since compactness is a very strong demand on a given set, many authors tried to relax it in order to obtain existence results in a less restrictive class of nonempty sets. In every case stronger conditions have to be placed on the coned. For instance, it is shown in [4] that if K is a closed convex cone in a Banach space E, satisfying the π-property (i.e. there exists a continuous functional f on E with $f(x) \leq 0$ for each $x \in K$ such that for every $\epsilon > 0$ the set $\{x \in K, -\epsilon \leq f(x)\}$, if nonempty, is relatively weakly compact in E) then $Min(A|K)$ is nonempty for every nonempty weakly closed set which is bounded from above. As for closed bounded sets, it should be remarked that the set of efficient points may be empty even when K is a closed convex pointed cone in a Banach space (see, e.g. [5]). However, Borwein [3] established that for such sets $Min(A|K)$ is nonempty if K is closed and Daniell and E is boundedly order complete. Recall that K is Daniell if any decreasing net having a lower bound converges to its infimum and E is boundedly order complete if any bounded decreasing net has an infimum (see [13]).

Some authors replaced the closedness and boundedness assumptions on A by the K-boundedness (that means there is a bounded set A_0 such that $A \subset A_0 + K$) and by the K-closedness (that means $A + clK$ is closed). For instance, some existence results in R^n [18] have been extended to a locally convex space E with the topology induced by a family $\{p_i: i \in \mathscr{I}\}$ of seminorms. Namely, in the case when E is quasicomplete and K is nuclear, the set of efficient points was shown to be nonvoid, firstly for a K-closed minorized (that means $A \subset a + K$ for some $a \in E$) [8] and later, for a K-closed K-bounded set [14]. Recall that E is quasicomplete if every closed bounded subset of E is complete and K is nuclear if for each $i \in \mathscr{I}$ there is a function f_i in the dual space E' such that $p_i(x) \leq f_i(x)$, $\forall x \in K$.

In this paper we study the existence of efficient points in a locally convex space for a nonempty set which is closed and bounded in the usual sense or with respect to a cone. Our results, Theorems 3.4 and 3.9, strengthen results of [3, 14] by weakening the hypothesis on the ordering cone yet maintaining the same hypothesis on the considered set. Our results are applicable not only to the case when K is not nuclear (which happens with the nonnegative orthants in the space L^p and Orlicz spaces) but also to cases when some results of [3, 9, 10] cannot be applied.

The paper is organized as follows. In Section 2 we introduce a new class of cones with two special properties. Section 3 is devoted to the existence of efficient points nonempty sets which are closed bounded in the usual sense or with respect to a cone.

2. Some Properties of Cones

We begin this section by recalling some definitions that will be used later. Our terminologies and notations are as in ([15] p. 215).

Let E be a locally convex space with the topology induced by a family $\{p_i: i \in \mathcal{J}\}$ of seminorms and ordered by a convex cone K. For a set $A \subset E$, $[A]$ means the set $(A+K) \cap (A-K)$. For a given filter $\mathcal{F}$, $[\mathcal{F}]$ denotes the filter $\{[\mathcal{F}]: F \in \mathcal{F}\}$. A net $\{x_\alpha: \alpha \in \mathcal{L}\}$ from E is called decreasing (increasing) if $x_\alpha \leqslant x_\beta (x_\beta \leqslant x_\alpha)$ for each α, $\beta \in \mathcal{L}$, $\beta \leqslant \alpha$ and monotone if it is decreasing or increasing.

We shall require the ordering cone K to have the following special properties.

DEFINITION 2.1. We say that K has property (*) if the set $(M+K) \cap (N-K)$ is bounded whatever nonempty bounded sets M and N are.

DEFINITION 2.2. We say that K has property (**) if one of the following equivalent conditions holds:

(i) Any bounded increasing net which is contained in K and in a complete subset of E has a limit;

(ii) Any bounded monotone net which is contained in a complete subset of E has a limit.

It is clear that if the condition (ii) is satisfied, then so is the condition (i). We shall show that if the condition (i) holds, then so does the condition (ii). Let $A = \{a_\alpha: \alpha \in \mathcal{L}\}$ is a bounded monotone net which is contained in a complete subset of E. We claim that the net A has a limit. Indeed, fix an index $\beta \in \mathcal{L}$ and consider the net $A' = \{a_\alpha - a_\beta: \alpha \geqslant \beta, \alpha \in \mathcal{L}\}$ (the net $A' = \{a_\beta - a_\alpha: \alpha \geqslant \beta, \alpha \in \mathcal{L}\}$) if A is increasing (decreasing). It is clear that the net A' is bounded, increasing and is contained in K and in a complete subset of E. Since the condition (i) is satisfied, the net A' has a limit and so does the net A.

REMARK 2.3. The reader can notice that if K' is a cone contained in K and K has property (*) and/or property (**), then so does K'.

Before giving some examples of cones with properties (*) and (**) let us recall the definition of normal cones, one of the most important notions in the theory of ordered spaces. As the reader will see later, any normal cone has property (*).

DEFINITION 2.4 ([15] p. 215). We say that K is normal if one of the following equivalent assertions holds:

(a) $\mathcal{U} = [\mathcal{U}]$ where $\mathcal{U}$ is a neighborhood filter of zero;

(b) For every filter $\mathcal{F}$ in E, $\lim \mathcal{F} = 0$ implies $\lim[\mathcal{F}] = 0$;

(c) There exists a generating family $\mathcal{P}$ of seminorms on E such that $p(x) \leqslant p(x+y)$ whenever x, $y \in K$ and $p \in \mathcal{P}$.

PROPOSITION 2.5. *Any normal cone has property* (*).

Proof. Let M, N be nonempty bounded sets in E and $u \in (M+K) \cap (N-K)$. Then there exist $m \in M$, $n \in N$, $k_1 \in K$, $k_2 \in K$ such that $u = m + k_1 = n - k_2$. Therefore,

$$0 \leqslant k_1 = u - m \leqslant k_1 + k_2 = n - m .$$

Let $\mathcal{P}$ be the family of seminorms in E as in (c) of Definition 2.4. Hence we have

$$0 \leqslant p(u-m) \leqslant p(n-m)$$

for each $p \in \mathcal{P}$. Consequently,

$$p(u) \leqslant p(u-m) + p(m) \leqslant p(n-m) + p(m) \leqslant p(n) + 2p(m) .$$

Make use of the boundedness of M and N, we conclude that $(M+K) \cap (N-K)$ is bounded, completing the proof.

COROLLARY 2.5.1 *Let K be a cone with a closed convex bounded base. Then K has property* (*).

Proof. First we show that K is normal. Let $\mathcal{F}$ be a filter in E convergent to zero and U is a neighborhood of zero. In view of ([10] Proposition 1.8, Chap. 1), there is a neighborhood of zero, say V, such that $[V] \subset U$. Since $\lim \mathcal{F} = 0$ we get $V \in \mathcal{F}$. Therefore, $[V] \in [\mathcal{F}]$ and, by the definition of a filter, $U \in [\mathcal{F}]$. Thus, $\lim[\mathcal{F}] = 0$ and the cone K is normal. The assertion follows from Proposition 2.5.

We now turn to property (**). It is easy to verify that if E is boundedly order complete and K is Daniell (see the definitions in Section 1) then K has this property. In order to give other examples of cones with property (**) we suppose that there is a family $\{f_i\colon i \in \mathcal{J}\}$ of functions from K into R_+ which can satisfy some of the following conditions:

C1. $f_i(x) \geqslant p_i(x)$ for all $x \in K$, $i \in \mathcal{J}$.

C2. For each $i \in \mathcal{J}$, $f_i(x)$ tends to zero as x tends to zero, $x \in K$.

C3. For each $i \in \mathcal{J}$, $f_i(.)$ is increasing, i.e. $f_i(x) \leqslant f_i(x+y)$ for each x, $y \in K$.

C4. Each function $f_i(.)$, $i \in \mathcal{J}$ maps any bounded subset of K into a bounded subset of R_+.

C5. If for some $i \in \mathcal{J}$, $p_i(x_j) \geqslant \epsilon > 0$ for $x_j \in K$, $j = 1, 2, \ldots$, then

$$\lim_{m\to\infty} f_i\left(\sum_{j=1}^{m} x_j\right) = \infty .$$

PROPOSITION 2.6. (1) *If conditions C1–C3 hold, then K has property* (*).
(2) *If conditions C3–C5 hold, then K has property* (**).

Proof. 1. Assume that conditions C1–C3 hold and M, N are nonempty bounded sets in E. Recall [15, p. 26] that a set M in E is bounded if and only if for arbitrary sequences $\{m_j\} \subset M$, $\{\lambda_j\} < R_+$, $\lambda_j \to 0$ the sequence $\lambda_j m_j$ tends to zero. Therefore, given a sequence $\{v_j\} \subset (M+K) \cap (N-K)$ with $v_j = m_j + k_j = n_j - k'_j$, $m_j \in M$, $n_j \in N$, $k_j \in K$, $k'_j \in K$, $j = 1, 2, \ldots$ it suffices to show that $\lim_{j\to\infty} \lambda_j v_j = 0$ for any nonnegative sequence $\{\lambda_j\}$ converging to zero. Since $\lambda_j(n_j - m_j) \to 0$ and by virtue of condition C2 we have $\lim_{j\to\infty} f_i(\lambda_j(n_j - m_j)) = 0$. On the other hand,

$$0 \leqslant f_i(\lambda_j - m_j)) = f_i(\lambda_j k_j) \leqslant f_i(\lambda_j(k_j + k'_j)) = f_i(\lambda_j(n_j - m_j))$$

for each $i \in \mathcal{J}$, $j = 1, 2, \ldots$ and, therefore, $f_i(\lambda_j(v_j - m_j))$ tends to zero as $j \to \infty$. By virtue of condition C1 and boundedness of M we get

$$\lim_{j\to\infty} p_i(\lambda_j v_j) = 0\,.$$

The last equation holds true for each $i \in \mathcal{J}$, hence $\lambda_j v_j$ tends to zero as j tends to ∞, as it was to be shown.

2. Assume the contrary that conditions C3–C5 hold but there is a bounded increasing net $\{x_l\} \subset K$ which is contained in a complete subset of E. Therefore, we can find a sequence $\{x_{l_k}\} \subset \{x_l\}$, $k = 1, 2, \ldots$ with $l_k \leqslant l_{k+1}$ such that $x_{l_{k+1}} - x_{l_k}$ does not converge to zero. It follows then that the existence of a positive scalar ϵ, an index $i_0 \in \mathcal{J}$ and a subsequence $\{x_{m_i}\} \leqslant \{x_{l_k}\}$ such that $p_{i_0}(x_{m_{i+1}} - x_{m_i}) \geqslant \epsilon$ for $i = 1, 2, \ldots$. By virtue of the monotonity of $\{x_l\}$ and condition C3 we obtain

$$\begin{aligned} f_{i_0}(x_{l_{k+1}}) &= f_{i_0}\Big(\sum_{j=1}^{k} (x_{l_{j+1}} - x_{l_j}) + x_{l_1}\Big) \\ &\geqslant f_{i_0}\Big(\sum_{j=1}^{k} (x_{l_{j+1}} - x_{l_j})\Big) \\ &\geqslant f_{i_0}\Big(\sum_{\{m_j\} \subset \{l_1, l_2, \ldots, l_k\}} (x_{m_{j+1}} - x_{m_j})\Big) \end{aligned}$$

which together with condition C5 gives

$$\lim_{k\to\infty} f_{i_0}(x_{l_{k+1}}) = \infty\,.$$

Thus f_{i_0} maps a bounded set $\{x_{l_k}\}$ into a unbounded set, contradicting condition C4.

Further we shall give the relationships between the just considered properties and the pointedness of a cone. Recall that K is pointed if $K \cap (-K) = \{0\}$ and acute if clK is pointed.

PROPOSITION 2.7. *K is pointed if one of the following conditions holds*:

(*a*) *K has property* (*);

(*b*) *K has a property* (**);

(*c*) *There is a family* $\{f_i\colon i \in \mathcal{I}\}$ *of nonnegative functions which are defined on K and satisfy condition C5.*

Proof. Assume the contrary that there exists a nonzero $x \in K \cap (-K)$.

In the first case, the set $\{x, 2x, \ldots\} \subset K \cap (-K)$ is unbounded, contradicting property (*) ($M = N = \{0\}$).

In the second case, the sequence $0 \leqslant x \leqslant -x \leqslant x \leqslant \ldots$ is monotone, bounded but does not converge to any point of E, a contradiction to property (**).

Finally, since E is Hausdorff, there exists $i_0 \in \mathcal{I}$ such that $p_{i_0}(x) = p_{i_0}(-x) > 0$. Denote $x_j = x$ if j is odd and $x_j = -x$ otherwise, we get

$$f_{i_0}\Big(\sum_{j=1}^{m} x_j\Big) = \begin{cases} f_{i_0}(x) & \text{if } m \text{ is odd}, \\ f_{i_0}(0) & \text{otherwise}, \end{cases}$$

contradicting condition C5.

PROPOSITION 2.8. *K is acute if one of the following conditions holds:*

(*a*) *K is normal;*

(*b*) *There is a family* $\{f_i\colon i \in \mathcal{I}\}$ *of nonnegative functions which are defined on K and satisfy conditions C1–C3.*

Proof. If K is normal, then *clK* is normal either [15, p. 216] and the assertion follows from Proposition 2.7.

Now assume that (b) holds but there are a nonzero x and nets $\{x_j\colon j \in \mathcal{L}\}$, $\{y_j\colon j \in \mathcal{L}\} \subset K$ satisfying $x = \lim x_j = \lim(-y_i)$. Without loss of generality we can assume that $p_{i_0}(x_j) \geqslant \gamma > 0$ for some positive scalar γ and $i_0 \in \mathcal{I}$. By virtue of conditions C1 and C3 we get

$$0 < \gamma \leqslant p_{i_0}(x_j) \leqslant f_{i_0}(x_j) \leqslant f_{i_0}(x_j + y_j)\,.$$

On the other hand, taking condition C2 into account we obtain $\lim f_{i_0}(x_j + y_j) = 0$, a contradiction.

Recall that K is said to be nuclear (or supernormal) if for each $i \in \mathcal{I}$ there is a functional f_i in the dual space E' such that $p_i(x) \leqslant f_i(x)$, $x \in K$. The following proposition shows that any nuclear cone has properties (*) and (**). Notice that the concept of nuclear cones defined in [1, 8] has many important applications in Pareto optimization in locally convex spaces, in the study of critical points for dynamical systems...

PROPOSITION 2.9. *Assume that K is nuclear. Then the family* $\{f_i\colon i \in \mathcal{I}\}$ *satisfies all conditions C1–C5 on both the cones K and clK and, therefore, these cones are normal, acute and have properties* (*), (**).

Proof. Observe that being in the dual space the functions f_i are linear and continuous. Hence they satisfy conditions C1–C4 on the whole *clK*. As for condition C5, assume that there are elements $x_j \in K$, $j = 1, 2, \ldots$ such that $p_i(x_j) \geqslant \epsilon > 0$ for some $i \in \mathcal{J}$. Make use of the linearity of f_i and the nuclearity of K we obtain

$$f_i\left(\sum_{j=1}^{m} x_j\right) = \sum_{j=1}^{m} f_i(x_j) \geqslant \sum_{j=1}^{m} p_i(x_j) \geqslant m\epsilon .$$

Thus conditions C5 holds on K. Analogously one can check that condition C5 holds on clK. The assertion follows from Propositions 2.6–2.8.

EXAMPLES 2.10. Let us give some examples of cones with properties (*) and (**).

Example 2.10.1. Any nuclear cone has properties (*) and (**). In particular, nonnegative orthant in R^n, the cone of nonnegative sequences in the Banach space l of summable sequences, the cone of nonnegative functions in the Banach space $C_{[0,1]}$ of continuous functions defined on $[0, 1]$ and the cone of (almost everywhere) nonnegative functions in the Banach space $L_{[0,1]}$ of integrable functions defined on $[0, 1]$ have properties (*) and (**).

Example 2.10.2. Let $L^p_{[0,1]}$, $1 < p < \infty$ be the Banach space of functions $x(.)$ on $[0, 1]$ which are integrable with respect to Lesbegue measure μ and $\int_0^1 |x(t)|^p \, d\mu < \infty$. Let K be the set of functions which are (almost everywhere) nonnegative. It was shown in [1, p. 98] that K has properties (*) and (**) but it is not nuclear.

Example 2.10.3 Let (T, Σ, μ) be a non-atomic positive measure space. By an Orlicz function we understand a nonzero mapping Φ: $R^1 \to [0, \infty]$ that is convex, even, vanishing and continuous at zero and left-continuous on the whole R_+. Let $F(\mu)$ denote the space of all (equivalence classes) μ-measureable functions x from T into R^1. Given an Orlicz function Φ, we define on $F(\mu)$ a convex functional I_Φ by

$$I_\Phi(x) = \int_T \Phi(x(t)) \, d\mu$$

and the Orlicz space $L^\Phi(\mu) = \{x \in F(\mu)\colon I_\Phi(\lambda x) < \infty$ for some $\lambda > 0\}$. This space equipped with the Luxemburg norm $\|x\|_\Phi = \inf\{\lambda > 0\colon I_\Phi(x/\lambda) \leqslant 1\}$ is a Banach space. Assume that the Orlicz function Φ satisfies the Δ_2-condition for all $u \in R^1$ (i.e. there exists a positive number η such that the inequality $\Phi(2u) \leqslant \eta\Phi(u)$ holds for all $u \in R^1$). Then the cone of nonnegative functions in this space has properties (*) and (**) but it is not necessarily nuclear [1]. The Banach space $L^\Phi(\mu)$ is not reflexive unless the dual function Φ' of Φ satisfies Δ_2-condition [12].

REMARKS 2.11.

1. In [1, 11] a class of so called completely correct cones (that means closed convex cones with property (**)) in Banach spaces has been investigated in detail.

In particular, it was shown that these cones have also property (*) and a cone K has property (**) if and only if the norm of E satisfies conditions C1–C5 on K.

2. Let $\mathscr{C}$ be a class of cones ensuring the existence of efficient points in a compact set in a topological vector space (see the definition in Section 1). Since $K \in \mathscr{C}$ whatever K is acute, Proposition 2.8 implies that if there is a family of functions $\{f_i: i \in \mathscr{I}\}$ satisfying C1–C3 on K (for example, if K is nuclear) then $K \in \mathscr{C}$.

3. Existence Theorems

Let A be a given nonempty set in a topological vector space E ordered by a convex cone K.

One of the most interesting questions in the theory of vector optimization is to study the conditions ensuring the existence of efficient points in a given set. Another question which is important in the theory of decision making is about the existence of an efficient alternative which is smaller (with respect to the ordering cone) than a given alternative. In the other words, for every point y in the given set A we would like to know whether there is an efficient point x such that $x \leqslant y$. That is the domination property (briefly denoted by (DP)) which first was introduced by Vogel and recently investigated in several works of Benson, Luc, Henig (see [10] and the references therein), Isac [8], Postolica [14].

The aim of this section is to give some sufficient conditions for a nonempty set in a locally convex space to have efficient points and the domination property. The sets under our investigation are assumed to be closed and bounded in the usual sense or with respect to a cone. By using the notions of cones with properties (*) and (**) considered in the previous section we shall improve some results of [3], [14].

Let us recall here some facts in the theory of existence of efficient points. For a set A and $x \in E$, A_x denotes the section $A \cap (x - K)$ of A at x. A set A is said to be K-complete if it has no cover of the form $\{(x_\alpha - clK)^C: \alpha \in \mathscr{L}\}$ with $\{x_\alpha: \alpha \in \mathscr{L}\}$ being a decreasing net in A. The reader interested in criterions of K-completeness is referred to ([10] Lemma 3.5, Chap. 2). We say that (DP) holds for A with respect to K if $A \subset Min(A|K) + K$.

THEOREM 3.1. ([10 Theorems 3.3 and 4.3, Chap. 2). *Assume that K is correct and A is nonempty set in E. Then*

(*i*) *$Min(A|K)$ is nonempty if and only if A has a nonempty K-complete section*;

(*ii*) *(DP) holds for A with respect to K if and only if for each $y \in A$, there is some $x \in A_y$ such that A_x is K-complete.*

PROPOSITION 3.2. (*i*) *If K is acute, then $Min(A|clK) \subset Min(A|K)$.*

(*ii*) *If K is acute, correct and (DP) holds for A with respect to clK, then (DP) holds for A with respect to K.*

Proof. (i) The assertion follows immediately from ([10] Proposition 2.4, Chap. 2).

(ii) Suppose that (DP) holds for A with respect to clK, i.e. $Min(A|clK)$ is nonempty and $A \subset Min(A|clK) + clK$. Observe that by virtue of the assertion (i), $Min(A|K)$ is nonempty. We have to show that (DP) holds for A with respect to K i.e. $A \subset Min(A|K) + K$. Noticing that

$$A = \{a\colon \exists b \in A \backslash \{a\}\colon a \in b + K\} \cup \{a\colon \nexists b \in A\backslash\{a\}\colon a \in b + K\}$$

we get

$$A \subset (A + K\backslash\{0\}) \cup Min(A|K) .$$

Since $A \subset Min(A|clK) + clK$, $Min(A|clK) \subset Min(A|K)$ and K is correct we obtain

$$\begin{aligned} A &\subset (A + K\backslash\{0\}) \cup Min(A|K) \\ &\subset (Min(A|clK) + clK + K\backslash\{0\}) \cup Min(A|K)) \\ &\subset (Min(A|K) + K\backslash\{0\}) \cup Min(A|K) \\ &= Min(A|K) + K , \end{aligned}$$

which concludes the proof.

In the remainder of this paper let E be a locally convex space with the topology induced by a family $\{p_i\colon i \in \mathcal{J}\}$ of seminorms and ordered by a convex cone K.

Now, using the notion of cones with property (**) we shall state a new condition for a set to be K-complete. The following proposition will play an important role in establishing the existence of efficient points.

PROPOSITION 3.3. *Assume that K has property* (**). *Then any bounded complete set A is K-complete.*

Proof. Let $\{x_\alpha\colon \alpha \in \mathcal{L}\}$ be a decreasing net in A. Making use of the boundedness and completeness of A and property (**) of K we conclude that $\{x_\alpha\colon \alpha \in \mathcal{L}\}$ converges to $x^* \in A$. Fix an arbitrary index α. Since $\{x_\alpha\colon \alpha \in \mathcal{L}\}$ is decreasing we have $x_\beta \leqslant x_\alpha$ for each $\beta \geqslant \alpha$ and, therefore, $x^* \in x_\alpha - clK$ for each $\alpha \in \mathcal{L}$. Thus $\{(x_\alpha - clK)^C\colon \alpha \in \mathcal{L}\}$ can not be a cover of A, as it was to be shown.

COROLLARY 3.3.1. *Let K be a correct cone with property* (**). *If there is an element $a \in A$ such that the section $A_a = A \cap (a - K)$ is bounded and complete, then $Min(A|K)$ is nonempty. If A_a is bounded and complete for every $a \in A$, then* (*DP*) *holds.*

Proof. An immediate consequence of Theorem 3.1 and Proposition 3.3.

COROLLARY 3.3.2. *Suppose that the cone clK has property (**). If there is an element* $\alpha \in A$ *such that the set* $A \cap (a - clK)$ *is bounded and complete, then* $Min(A|K)$ *is nonempty. If in addition K is correct and* $A \cap (a - clK)$ *is bounded and complete for each* $a \in A$*, then (DP) holds for A with respect to K.*

Proof. Observe that by Proposition 2.7 the cone K is acute. The assertion follows from Corollary 3.3.1 and Proposition 3.2.

As an immediate consequence of Corollary 3.3.2 we have the following

COROLLARY 3.3.3. *Suppose that the cone clK has property (**) and A is complete and bounded. Then* $Min(A|K)$ *is nonempty. If in addition K is correct, then (DP) holds for A with respect to K.*

Recall that E is quasicomplete if any closed bounded subset of E is complete. Using the notion of quasicompleteness we can derive from the Corollaries 3.3.1–3.3.3 some facts about the existence of efficient points for sets in quasicomplete sets. For example, we have the following

COROLLARY 3.3.4. *Suppose that E is quasicomplete and the cone clK has property (**). If there is an element* $a \in E$ *such that* $A \cap (a - clK)$ *is closed and bounded, then* $Min(A|K)$ *is nonempty. If* $A \cap (a - clK)$ *is closed and bounded for each* $a \in A$ *and K is correct, then (DP) holds for A with respect to K.*

Our main theorem about the existence of efficient points in closed bounded sets reads as follows

THEOREM 3.4. *Suppose that E is quasicomplete and the cone clK has property (**). Then for any nonempty closed bounded set A,* $Min(A|K)$ *is nonempty. If K is correct, then (DP) holds for A with respect to K.*

Proof. An immediate consequence of Corollary 3.3.4.

REMARKS 3.5. We would like to say some words on the existence of efficient points in a nonempty closed bounded set. As it was shown [5] such a set may have not any efficient point even when it is in a Banach space with a closed convex pointed cone. A question worthy of interest is under which conditions a nonempty closed bounded set has efficient points. A result due to Borwein [3] shows that if K is closed these conditions are the boundedly order completeness of E and the Daniell property of K. Theorem 3.4 is a new existence result without the closedness assumption on K and it can be applied, for instance, to the cone K_0 of positive functions in $L^p_{[0,1]}$, $C_{[0,1]}$. Further, it should be remarked that neither the closedness and boundedness of A nor property (**) of the cone K can be dropped. To see this let us consider the following examples.

Example 3.5.1. Let K be the nonnegative orthant in R^n, $A = -K$ and $B = \{x \in$

K, $0 < \|x\| \leq 1\}$. The closed set A is unbounded, the bounded set B is not closed and neither A nor B has efficient points.

Example 3.5.2. (Sterna-Karwat [16], see also [10] p. 51). Let Ω be the vector space of all sequences $x = \{x_m\}$ such that $x_m = 0$ for all but a finite number of choice for m. It is a quasicomplete space if we provide it with the norm

$$\|x\| = \max\{|x_m|: m = 1, 2, \ldots\} .$$

Let K be the cone composed of zero and of sequences whose last nonzero term is positive. Holmes called such a cone ubiquitous. This cone has neither property (*) nor property (**). Now let e_n stay for the vector with the unique nonzero components being 1 at the n-place. Consider the set

$$A = \{x_0\} \cup \left\{ \bigcup_{n=1}^{\infty} \sum_{i=1}^{n} x_i : n = 1, 2, \ldots \right\},$$

where

$$x_0 = e_1$$

$$x_n = \sum_{i=1}^{n-1} e_i/2^{n-1} - e_n/2^{n-1}, \quad n \geq 1 .$$

Then A is compact because

$$\lim_{n\to\infty} \sum_{i=1}^{n} x_i = x_0 .$$

Furthermore,

$$x_0 > \sum_{i=1}^{n} x_i > \sum_{i=1}^{n+1} x_i$$

which shows that $Min(A|K)$ is empty.

PROPOSITION 3.6. *Assume that*

(*i*) *K is correct and has property* (**);

(*ii*) *For a nonempty set A in E there is an element $a \in E$ such that $(A + K) \cap (a - K)$ is bounded and complete.*

Then $Min(A|K)$ is nonempty. If for every $a \in A$ the set $(A + K) \cap (a - K)$ is bounded and complete then (DP) holds.

Proof. Firstly remark that in view of Proposition 2.6 K is pointed. Further, setting $A' = (A + K) \cap (a - K)$, $A'_a = A' - a$ we get $0 \in A'_a$ and $A'_a = A'_a \cap (-K)$. Therefore, A'_a is a section of A'_a at 0. Since A' is bounded complete, this section is bounded complete and by virtue of Proposition 3.3, it is K-complete. Theorem 3.1 applied to the set A' shows that $Min(A'|K)$ is nonempty.

The remainder of this proof is as in the proof of Theorem 1 in [14]. Firstly, we show that $Min(A'|K) \subset Min(A|K)$. Indeed, if x is an arbitrary element in $Min(A'|K) \backslash A$, then by the definition of A' there exist $a_0 \in A$ and k, $k_1 \in K \backslash \{0\}$ such that $x = a_0 + k = a - k_1$, that is $a_0 \leqslant x$. On the other hand, $a_0 = a - (k + k_1) \subset (A + K) \cap (a - K) = A'$ and $a_0 < x$, a contradiction. Hence $Min(A'|K) \subset A$. Suppose now that $x \in Min(A'|K) \backslash Min(A|K)$. Then there is $a_1 \in A$ such that $x - a_1 \in K \backslash \{0\}$. Consequently, $a_1 \in x - K \subset a - K$ and $a_1 \in A \subset A + K$, that means $a_1 \in A'$, contradicting the fact that x is an efficient point of A'. Therefore, $Min(A'|K) \subset Min(A|K)$ and $Min(A|K)$ is nonempty.

Suppose now that for every $a \in A$, the set $A' = (A + K) \cap (a - K)$ is bounded and complete. Then, as it was just shown, there is an element $a_0 \in Min(A'|K) \subset Min(A|K)$. By the definition of A', $a \in a_0 + K$. Hence $a \in a_0 + K \subset Min(A'|K) + K \subset Min(A|K) + K$ and (DP) holds. Thus, the theorem is proven.

We are ready now to formulate the existence theorems for nonempty sets which are closed and bounded with respect to the ordering cone. We say that A is K-closed [10] if $A + clK$ is closed and A is K-bounded [8, 10] if there is a nonempty bounded set A_0 such that $A \subset A_0 + K$. It is clear that any bounded set is K-bounded but the cone K being K-bounded is unbounded. Let $A = [0, 1) \subset R^1$, $K = [0, \infty)$ then the set A is K-closed but it is not closed. The following examples show that the class of K-closed K-bounded sets does not coincide with the class of closed bounded sets.

EXAMPLES 3.7.

Example 3.7.1. Let E be the Banach space $C_{[0,1]}$ of continuous functions defined on $[0, 1]$, K be the cone of nonnegative increasing functions and $A = -K$. In view of Weierstrass' theorem, any continuous function can be approximated by polynomials while the later are representable as differences of nonnegative increasing functions. Thus, $E = cl(K - K)$. On the other hand, the function $x(.)$ defined by $x(t) = t\cos(\pi/2t)$ for $t \neq 0$ and $x(0) = 0$ can not be represented as a difference of nonnegative increasing functions. Therefore, $K - K \neq E$ and $K - K$ is not closed, that means the closed set $A = -K$ is not K-closed. Now let $A = \{x \in -K, \|x\| \leqslant 1\}$. By the same argument as above, one can verify that the closed bounded set A is not K-closed.

Example 3.7.2. Let E be the Orlicz space $L^{\Phi}(\mu)$ whose function Φ satisfies Δ_2-condition and K be the cone of nonnegative functions as in Example 2.10.3. Let C be a convex compact set such that $0 \in C$ and $\exists x_0 \in C \cap K$, $\|x_0\| > 1$. Let $A = \{x \in K \cap C, \|x\| \geqslant 1\} \backslash \{tx_0, t \in (0, 1)\}$. It is clear that A is bounded but is not closed. We claim that A is K-closed. Indeed, given a sequence $(a_m + k_m)_{m \geqslant 1}$ converging to b with $a_m \in A$, $k_m \in K$. Since A is contained in a compact set C, we can assume that a_m converges to $a \in C$ and k_m converges to $k \in K$. If $a \in A$ the proof is finished. Assume that $a \in C \backslash A$. Then $a \in \{tx_0, t \in (1, \infty)\}$ and $a = t_0 x_0$ with $t_0 > 1$. Since $b = a + k$ we have $b = t_0 x_0 + k = x_0 + (t_0 - 1)x_0 + k$. By the assumption, we get $(t_0 - 1)x_0 + k \in K$ and therefore, $b \in A + K$. Thus A is K-closed.

THEOREM 3.8. *Suppose that E is quasicomplete and K is a closed convex cone with properties* (*) *and* (**). *Then for any nonempty K-closed K-bounded set A, Min(A|K) is nonempty and (DP) holds.*

Proof. Let $a \in A$ be an arbitrary element. Consider the set $A' = (A + K) \cap (a - K)$. It is clear that A' is closed. Since A is K-bounded, there is a bounded set A_0 in E such that $A \subset (A_0 + K)$. Therefore, $A' \subset (A_0 + K) \cap (a - K)$ and, by virtue of property (*), it is bounded too. In view of the quasicompleteness of E the set A' is then complete. All conditions of Proposition 3.6 are satisfied and the assertion follows.

THEOREM 3.9. *Suppose that E is quasicomplete and the cone clK has properties* (*) *and* (**). *Then for any nonempty K-closed K-bounded set A, Min(A|K) is nonempty and, if in addition K is correct, then (DP) holds for A with respect to K.*

Proof. Observe first that being K-closed and K-bounded the set A is clK-closed and clK-bounded. Theorem 3.8 applied to the set A and the cone clK implies that $Min(A|clK) \neq \emptyset$ and $A \subset Min(A|clK) + clK$. In view of Proposition 2.7, the cone K is acute and the assertion follows from Proposition 3.2.

As an immediate consequence we have the following

COROLLARY 3.9.1 ([14] Theorem 1). *Let K be a closed nuclear cone in a quasicomplete space E. Then for any K-closed K-bounded set, Min(A|K) nonempty and (DP) holds.*

REMARKS 3.10.

1. The assumptions about the K-closedness and K-boundedness of A and properties (*), (**) of K can not be dropped. Indeed, consider the following examples. Let K be the nonnegative orthant in R^n, $A = K\backslash\{0\}$, $B = -K$. Then the K-bounded set As is not K-closed, the K-closed set B (since $B + K = -K + K = R^n$ is closed) is unbounded and neither A nor B has efficient points. The set A in Example 3.5.2 being compact is a K-closed K-bounded set in a quasicomplete space but it has not any efficient point since the cone K has neither property (*) nor property (**).

2. Let E, K, A be as in Example 3.7.2. By virtue of Theorem 3.9, the set of efficient points of A is nonvoid and the domination property holds for this set with respect not only to K but also to K_0, where K_0 is the cone of positive functions. The reader can notice that some results of Borwein, Jahn, Luc (see [10] Chap. 2) and Postolica [14] cannot be applied to this case since the assumptions on the closedness (of A or K), on the reflexivity (of E) and on the nuclearity (of K) are not satisfied.

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